The effective action for a relativistic Jaynes-Cummings model

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Abstract

We describe an effective field theory for atomic lasers which reduces to the Jaynes-Cummings model in the non-relativistic, single mode limit. Our action describes a multi-mode system, with general polarizations and Lorentz invariance and can therefore be used in all contexts from the astrophysical to the laboratory. We show how to compute the effective action for this model and perform the calculation explicitly at the one loop level. Our model provides a way of analyzing a many-particle, two-state model with arbitrary boundary conditions.

I. INTRODUCTION

The canonical model for laser physics is the Jaynes-Cummings model. It describes a single mode oscillator representing a coherent electromagnetic field, coupled to a two level reservoir of atoms [1]. The Jaynes-Cummings model is defined in momentum space in terms of the photon creation and annihilation operators a and $a^\dagger$ for a single momentum mode $K = \Omega/c$ and a single, unspecified polarization of the electromagnetic field. The quantum mechanical Hamiltonian is given by

$$H = a^\dagger a \hbar\Omega + \frac{1}{2}\hbar\omega_{12}\sigma_z + \hbar g(\sigma_+ a + a^\dagger \sigma_-), \tag{1}$$

where $\hbar\omega_{12}$ is the energy difference between the atomic states. The creation and annihilation operators satisfy $[a, a^\dagger] = 1$ and the sigma matrices satisfy $[\sigma_+, \sigma_-] = \sigma_z$. This model is the natural candidate for studying the fundamentals of the interaction between matter and radiation in a laser in a wide variety of situations, but it has several shortcomings and it is important to understand how these may be resolved in a reasonable fashion.

In this paper we present a new model which retains the essential simplicity of the Jaynes-Cummings model, but which repairs some of its limitations. One of our principal aims is to write down a real-space Lagrangian formulation for a two-state system in which spectral content and polarizations are fully general: this should not only allow us to use the full machinery of modern field theory with all its attendant methodology, but also admit the solution of problems with general boundary conditions, in contact with many particle reservoirs. The theory makes gauge symmetries and the space-time structure clearer and leaves us free to use well-established path integral or Green function methods for computing the effective action. Finally, but not least importantly, it also bridges a cultural gap between the worlds of field theory and laser physics.

The Jaynes-Cummings model is an idealized description of laser phenomena. As a single-mode theory it can not address boundary conditions [2,3] or time-dependent interactions [4] since, by the uncertainty principle, a single mode must be completely delocalized in space

and time. These are features characteristic of the micro-maser and of non-linear media. Almost all of the work on lasers is phenomenological and couched in momentum space. Real space methods were pioneered by Graham and Haken [5–7], but the closest attempts at constructing a microscopic description of the laser come from Korenman's [8] use of Schwinger's action formulation [9]. A recent letter makes some progress with this approach for semiconductor lasers [10]. These papers also deal with effective theories however. Our paper is no different in this respect: we use an effective interaction and effective field variables. Indeed it would be inconceivable to attempt to write down a theory in which every optically-active electron and background charge were dealt with explicitly. Rather we pose the question: what are the relevant degrees of freedom for the laser at the energy scales of interest? These are clearly the averaged atomic properties and the magnitude of electromagnetic field.

A disadvantage with Korenman's analysis is his use of non-relativistic field theory. Korenman begins with the Schrödinger equation coupled to reservoirs and seeks self-consistent solutions for decays rates and line widths. But radiative corrections to the non-relativistic theory are beset with problems: acausal loop diagrams, such as those used in constructing the effective action, vanish owing to the absence of anti-particles (negative energy states) in the non-relativistic theory. This makes the non-relativistic theory alien to field theorists who are used to the language of Feynman diagrams and Green functions and, in any case, one would expect a physical system described by the Schrödinger equation to arise naturally from a more general relativistic theory in the low energy limit. There is then the issue of non-renormalizability: Schrödinger scalar field theory is more divergent than relativistic scalar field theory, owing to the dimension of the field variables, and is specifically non-renormalizable in $3+1$ dimensions. It therefore makes more sense to begin with a relativistic theory, which is renormalizable, and consider the non-relativistic theory as an approximation to this full theory. In addition, we expect that a relativistic theory is necessary to study astrophysical situations, where the motion of atoms could be relativistic at high temperatures, even when the emitted radiation is of low energy. We begin therefore by introducing the action for a relativistic two-state model.

II. THE ACTION AND ITS INTERPRETATION

Consider a system of neutral atoms, containing optically-active electrons, which endow the atoms with a dipole moment. The electrons will not be explicit degrees of freedom in our model, rather their presence will be taken into account by the availability of transitions between the two atomic states. A neutral atoms is therefore represented as a two component real-scalar field; the two components represent the lower (unexcited) and upper (excited) levels of the atom. Each level has a different effective 'mass', in relativistic terminology $m_a = m + E_a/c^2$, where m is the atomic mass and the potential-energy of the level is E_a. In SI units, the action has the following form:

$$S = \int dV_x \left\{ \frac{1}{2}\hbar^2 c^2 (\partial^\mu \phi_a)(\partial_\mu \phi_a) + \frac{1}{2} m_a^2 c^4 \phi_a \phi_a + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + P^{\mu\nu}(\phi) F_{\mu\nu} \right\} \tag{2}$$

where $a = 1, 2$ and $P^{\mu\nu}(\phi)$ is a polarization tensor which is to be specified below. Our conventions are such that the Minkowski metric tensor $g_{\mu\nu}$ has the signature $-+++$ and we use symbols $d\sigma_x$ to represent an n-dimensional infinitesimal spatial volume element on a spacelike hypersurface and dV_x to represent the $n+1$-dimensional infinitesimal spacetime volume which is canonically written $d\sigma_x dx^0 \sqrt{-\det g}$. In other words, σ_x is a spatial volume and V_x is a spacetime volume.

The essential physics of this model is determined by the form of the dipole interaction tensor $P^{\mu\nu}(\phi)$. Given that the dynamical degrees of freedom are represented by real scalar fields, we have only two choices for this quantity, as we discuss below. The form for such a dipole term is unfamiliar in a relativistic theory, so we allow ourselves to be guided by the non-relativistic limit and require that this limit be consistent with known results, namely the non-relativistic analysis of Korenman [8] and in turn the Jaynes-Cummings model [1]. In particular, in the non-relativistic limit, one should obtain an expression for $P_{\mu\nu}$ of the form used by Korenman:

$$P_{\mu\nu} \to \gamma_{\mu\nu}^{ab} \psi_a^* \psi_b, \tag{3}$$

for some constant, off-diagonal matrix $\gamma^{ab}_{\mu\nu}$. It is evident that this is a dipole induced transition from the form of the operators. ψ^* is a creation operator for the field and ψ is a destruction operator, thus the off-diagonal operator creates an upper state and destroys a lower state, or vice-versa. Moreover, the components γ_{0i} of this matrix will be proportional to the electric dipole moment of the atom. One relativistic generalization which reduces to eqn. (3), is

$$\tilde{P}_{\mu\nu} = i\hbar\gamma_{\mu\nu}\tilde{\epsilon}^{ab}\phi_a\partial_0\phi_b, \tag{4}$$

where $\gamma_{\mu\nu}$ is a constant, anti-symmetric tensor and $\tilde{\epsilon}_{ab}$ is the two-dimensional antisymmetric Levi-Civita symbol [11]. This form is intuitively appealing because it seems to be related to the relativistic inner product:

$$(\phi_a, \phi_b) = i\hbar c^2 \int d\sigma_x \frac{1}{2}(\phi^*_a\partial_0\phi_b - (\partial_0\phi^*_a)\phi_b). \tag{5}$$

Unfortunately, this form for $\tilde{P}_{\mu\nu}$ raises some questions concerning renormalizability (see section V). It is non-renormalizable in $3+1$ and $2+1$ dimensions: in particular we expect new, higher derivative interactions to be introduced at each order in perturbation theory. Although low energy predictions are still possible in such theories, we avoid this problem by introducing another interaction

$$P_{\mu\nu} = \frac{1}{2}\gamma_{\mu\nu}\overline{\epsilon}^{ab}\phi_a\phi_b, \tag{6}$$

where $\gamma_{\mu\nu}$ is antisymmetric in μ and ν and

$$\overline{\epsilon}^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is now *symmetric* in a and b. In the non-relativistic limit, this interaction differs from that in eqn. (4) only by a factor of $i\hbar/mc^2$ (introduced by ∂_t). It has the advantage of being marginally renormalizable in $3+1$ dimensions and super-renormalizable in $2+1$ dimensions. This will be discussed in more detail in Section V, in which the one loop effective action is computed explicitly.

We now sketch a simple derivation of the non-relativistic limit for our action, in order to give a physical interpretation to the constant matrix $\gamma_{\mu\nu}$. The next section contains a further justification of this method based on the field equations. The limiting procedure is unambiguous up to redefinitions of the origin for the arbitrary energy scale. The simplest procedure is to first observe that the real scalar field $\phi(x)$ may be decomposed into

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \tag{7}$$

where $\phi^{(+)}(x)$ is the positive frequency part of the field and $\phi^{(-)}(x)$ is the negative frequency part of the field and $\phi^{(+)}(x) = (\phi^{(-)}(x))^*$. We now rescale the fields by the atomic mass:

$$\phi^{(+)}(x) = \frac{\psi(x)}{\sqrt{2mc^3}} \quad , \qquad \phi^{(-)}(x) = \frac{\psi^*(x)}{\sqrt{2mc^3}} \tag{8}$$

In addition, we note that the relativistic energy operator $i\hbar\partial_t$ is related to the non-relativistic energy operator $i\hbar\tilde{\partial}_t$ by a shift with respect to the rest energy of particles:

$$i\hbar\partial_t = mc^2 + i\hbar\tilde{\partial}_t. \tag{9}$$

This is because the non-relativistic Hamiltonian does not include the rest energy of particles, its zero point begins just above the rest energy.

Integrating the kinetic term by parts so that $(\partial_\mu\phi)^2 \to \phi(-\Box)\phi$ and substituting eqn. (8) into eqns. (2) and (6) gives,

$$\begin{aligned} S = &\int d\sigma_x dt \frac{1}{2}(\psi+\psi^*)_a \left\{ \frac{\hbar^2\tilde{\partial}_t^2}{mc^2} - i\hbar\tilde{\partial}_t + \frac{E_a^2}{2mc^2} + E_a - \frac{\hbar^2}{2m}\nabla^2 \right\} (\psi+\psi^*)_a \\ &+ \int d\sigma_x dt \frac{\gamma^{\mu\nu}\overline{\epsilon}_{ab}}{4mc^2} F_{\mu\nu}(\psi+\psi^*)_a(\psi+\psi^*)_b. \end{aligned} \tag{10}$$

Here we have dropped the Maxwell part of the action to avoid clutter, since it has no non-relativistic limit. If we use the fact that $\psi_a(x)$ is composed of only positive plane-wave frequencies, it follows that terms involving ψ^2 or $(\psi^*)^2$ vanish since they involve delta functions imposing a non-satisfiable condition on the energy $\delta(mc^2 + \hbar\tilde{\omega})$, where both m and $\tilde{\omega}$ are greater than zero. This assumption ceases to be true only if there is an explicit time-dependence in the action, indicating a non-equilibrium scenario, or if the mass of the

atoms goes to zero (in which case the NR limit is unphysical). In the next section we perform a transformation of the field equations which decouples the positive and negative frequency modes, justifying this procedure in a more conventional way. We are therefore left with

$$S_{NR} = \lim_{c\to\infty} \int d\sigma_x dt \left\{ \frac{i\hbar}{2} \left(\psi_a^*(\tilde{\partial}_t \psi_a) - (\tilde{\partial}_t \psi_a^*)\psi_a \right) - \psi_a^* H_a \psi_a - \frac{\gamma^{\mu\nu}\overline{\epsilon}_{ab}}{4mc^2} F_{\mu\nu}(\psi_a^*\psi_b + \psi_a\psi_b^*) \right\} \tag{11}$$

where the differential operator H_a is defined by

$$H_a = -\frac{\hbar^2\nabla^2}{2m} + E_a + \frac{1}{2mc^2}(E_a^2 + \tilde{\partial}_t^2), \tag{12}$$

and we have redefined the action by a sign in passing to a Euclideanized non-relativistic metric. It is now clear that, in the NR limit $c \to \infty$, the final two terms in H_a become negligable, leading to the field equation

$$H_a\psi_a(x) + \frac{\gamma^{\mu\nu}\overline{\epsilon}^{ab}}{2mc^2} F_{\mu\nu}\psi_b(x) = i\hbar\tilde{\partial}_t\psi_a(x), \tag{13}$$

which is the Schödinger equation of a particle of mass m moving in a constant potential of energy E_a with a dipole interaction. The dipole interaction term is not negligeable since the constant $\gamma^{\mu\nu}$ is of order c^3 as we shall show below.

The space-time components γ^{0i} can now be related to physical electric dipole moments for linear media in the following manner. From classical electromagnetism we have that the dipole energy density is given by $\mathbf{P}\cdot\mathbf{E}$, where $\mathbf{P}$ is the dielectric polarization and $\mathbf{E}$ is the electric field. The dielectric polarization is related to microscopic displacements of charge by

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = -\langle e\mathbf{r}\rangle \times \text{no. density of charges.} \tag{14}$$

If we use the quantum number-density $\psi^*\psi$ here we see that the dipole energy density is given by

$$\mathbf{P}\cdot\mathbf{E} = -\psi^*\psi\ \langle e\mathbf{r}\rangle \cdot \mathbf{E}. \tag{15}$$

Our non-relativistic Lagrangian is also an energy density, thus comparing these in the rest frame of the charges, and using the fact that $F^{0i} = -E^i/c$ we have

$$\frac{\gamma^{\mu\nu}F_{\mu\nu}}{2mc^2}\psi^*\psi = \frac{\gamma^{0i}}{mc^2}F_{0i}\psi^*\psi = -\frac{\gamma^{0i}}{mc^3}E_i\psi^*\psi = -\langle e\mathbf{r}\cdot\mathbf{E}\rangle\psi^*\psi, \tag{16}$$

allowing us to identify

$$\gamma^{0i} = mc^3\langle er^i\rangle. \tag{17}$$

Note that m is the mass of an atom and not the mass of the polarized charges. The spatial components γ^{ij} are normally zero in the laboratory frame, but in relatively moving frames they may be determined by a suitable boost transformation.

III. RELATIONSHIP TO THE JAYNES-CUMMINGS MODEL

We now wish to show rigorously, making explicit the dimensionless parameters that have to be small in order for the approximation to work that, in the non-relativistic limit, our model describes a two level atomic system interacting via a dipole interaction with an electromagnetic field. For atoms interacting with a single radiation mode, the Jaynes-Cummings model emerges naturally. The Lagrangian for our model is

$$-\mathcal{L} = \frac{1}{2}\hbar^2c^2(\partial_\mu\phi_a)^2 + \frac{1}{2}m_a^2c^4\phi_a^2 + \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \gamma^{\mu\nu}\overline{\epsilon}_{ab}\phi_a\phi_b\partial_\mu A_\nu. \tag{18}$$

which gives for the equation of motion,

$$\left(-\Box + \frac{m_a^2c^2}{\hbar^2}\right)\phi_a + \frac{2}{\hbar^2c^2}\overline{\epsilon}_{ab}\phi_b\gamma^{\mu\nu}\partial_\mu A_\nu = 0. \tag{19}$$

To take the non-relativistic limit we define two fields [12],

$$\begin{aligned}\psi_a &= \sqrt{\frac{m_ac^2}{2}}(\phi_a + \frac{i\hbar}{m_ac^2}\dot{\phi}_a)\\ \chi_a &= \sqrt{\frac{m_ac^2}{2}}(\phi_a - \frac{i\hbar}{m_ac^2}\dot{\phi}_a).\end{aligned} \tag{20}$$

The rescaling is necessary in order that the non-relativistic wave-functions have the right dimensions, with standard inner product. Clearly, if ϕ_a is real, then $\psi_a = \chi_a^*$. Moreover, these definitions imply:

$$\phi_a = \frac{1}{\sqrt{2m_a c^2}}(\psi_a + \chi_a) \tag{21}$$

$$i\dot{\phi}_a = \left(\frac{m_a c^2}{\hbar}\right)\frac{1}{\sqrt{2m_a c^2}}(\psi_a - \chi_a) \tag{22}$$

from which one can deduce:

$$i(\dot{\psi}_a + \dot{\chi}_a) = \frac{m_a c^2}{\hbar}(\psi_a - \chi_a) \tag{23}$$

This field redefinition reduces the action to one that is first order in time derivatives. Using standard Legendre transform theory, we obtain the part of the Hamiltonian density involving the scalar fields

$$\begin{aligned} H(\pi_a, \phi_a) &= \pi_a\dot{\phi}_a - \mathcal{L} \\ &= \frac{1}{2c^2\hbar^2}\pi_a^2 + \frac{1}{2}\hbar^2 c^2(\nabla\phi_a)^2 + \frac{1}{2}m_a^2\phi_a^2 + \frac{1}{2}\gamma\cdot F\epsilon_{ab}\phi_a\phi_b \end{aligned} \tag{24}$$

where

$$\pi_a = c\frac{\delta\mathcal{L}}{\delta(\dot{\phi}_a)} = c\hbar^2\dot{\phi}_a = -ic\hbar\sqrt{\frac{m_a c^2}{2}}(\psi_a - \chi_a) \tag{25}$$

and we have written $\gamma\cdot F \equiv \gamma^{\mu\nu}F_{\mu\nu}$ for brevity. Replacing π_a and ϕ_a by their definitions in terms of ψ_a and χ_a as given above, we get the Hamiltonian density,

$$H(\psi_a, \chi_a) = m_a c^2\psi_a\chi_a + \frac{\hbar}{4m_a}(\partial_i(\psi_a + \chi_a)\partial_i(\psi_a + \chi_a)) + \frac{\gamma\cdot F}{4\sqrt{m_a m_b}}\epsilon_{ab}(\psi_a + \chi_a)(\psi_b + \chi_b) \tag{26}$$

and

$$\begin{aligned} \mathcal{L} &= \pi_a\dot{\phi}_a - H(\pi_a, \phi_a) \\ &= i\hbar\chi_a\dot{\psi}_a - m_a c^2\psi_a\chi_a \\ &\quad -\frac{\hbar}{4m_a}(\partial_i(\psi_a + \chi_a)\partial_i(\psi_a + \chi_a)) - \frac{\gamma\cdot F}{4\sqrt{m_a m_b}}\epsilon_{ab}(\psi_a + \chi_a)(\psi_b + \chi_b) \end{aligned} \tag{27}$$

up to total derivatives. We obtain the equations of motion by varying with respect to ψ_a and χ_a:

$$\begin{aligned} i\hbar\dot{\psi}_a &= -\frac{\hbar^2}{2m_a}\nabla^2(\psi_a + \chi_a) + m_a c^2\psi_a + \frac{\epsilon_{ab}\gamma\cdot F}{2c^2\sqrt{m_a m_b}}(\psi_b + \chi_b) \\ i\hbar\dot{\chi}_a &= \frac{\hbar^2}{2m_a}\nabla^2(\psi_a + \chi_a) - m_a c^2\chi_a - \frac{\epsilon_{ab}\gamma\cdot F}{2c^2\sqrt{m_a m_b}}(\psi_b + \chi_b) \end{aligned} \tag{28}$$

and write the Hamiltonian as a four by four matrix in the 2-component space $\{a, b\}$ crossed into the 2-component space $\{\psi, \chi\}$:

$$H_{ab} = \left[\delta_{ab}\left(-\frac{\hbar^2\nabla^2}{2m_a} + m_a c^2\right) + \frac{\gamma \cdot F}{2c^2\sqrt{m_a m_b}}\epsilon_{ab}\right]\beta + \left[\delta_{ab}\left(-\frac{\hbar^2\nabla^2}{2m_a}\right) + \frac{\gamma \cdot F}{2c^2\sqrt{m_a m_b}}\epsilon_{ab}\right]\mathcal{O} \tag{29}$$

where

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \mathcal{O} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{30}$$

$$\tag{31}$$

So far these equations are exact. Note that the terms proportional to $\mathcal{O}$ are non-Hermition and couple positive and negative energy states. We want to perform a similarity transformation (non-unitary) that will remove the operator $\mathcal{O}$ that couples ψ_a and χ_a. We use an operator of the form $U_F = e^{i\Lambda}$ where Λ has no explicit time dependence. We will not be able to find the required Λ exactly, so we assume that a perturbative expansion exists in which Λ is small. In this case, $\phi' = e^{i\Lambda}\phi$ and $\phi' = H'\phi'$ which gives,

$$\begin{aligned} H' &= e^{i\Lambda} H e^{-i\Lambda} \\ &= H + i[\Lambda, H] + \ \ldots \end{aligned} \tag{32}$$

We will in fact need three independent expansion parameters. In addition to the usual non-relativistic expansion parameters λ_a for the two atomic states:

$$\lambda_a = \frac{\hbar^2\nabla^2}{m_a^2 c^2} \tag{33}$$

we will need the coupling expansion parameter:

$$\lambda_3 = \frac{\gamma \cdot F}{\overline{m}c^2\sqrt{m_1 m_2 c^2}} \tag{34}$$

where $\overline{m} = \frac{1}{2}(m_1 + m_2)$. We will also assume that $(m_1 - m_2) << \overline{m}$. Since, as argued above, $\gamma \cdot F \propto mc^2 e\langle \mathbf{r} \cdot \mathbf{E}\rangle$, it follows that

$$\lambda_3 \sim \frac{e\langle \mathbf{r}\cdot\mathbf{E}\rangle}{\overline{m}c^2} \tag{35}$$

Thus $\lambda_3 << 1$ requires the dipole energy in the electric field to be much smaller than the rest energy of the atom.

We can now expand Λ:

$$\Lambda = \Lambda_1 + \Lambda_2 + O(\lambda^2) \tag{36}$$

where $O(\lambda^2)$ refers to a product of any two of the small expansion parameters, and

$$\Lambda_1 = \frac{i}{2}\lambda_a \delta_{ab}\beta\mathcal{O} \tag{37}$$

$$\Lambda_2 = -\frac{i}{2}\lambda_3 \epsilon_{ab}\beta\mathcal{O} \tag{38}$$

In the above

$$\beta\mathcal{O} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{39}$$

It is easy to verify that $i[\Lambda, H]$ to leading order in λ exactly cancels the terms in H that are proportional to $\mathcal{O}$. This decoupling of the ψ and χ modes yields a Hamiltonian which can be written as a two by two matrix in the space {a,b} acting on the column vector

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

In particular (dropping the prime):

$$H = H_0 + H_{int} \tag{40}$$

where the free part of the Hamiltonian is:

$$H_0 = \begin{pmatrix} -\frac{\hbar^2\nabla^2}{2m_1} + m_1c^2 & 0 \\ 0 & -\frac{\hbar^2\nabla^2}{2m_2} + m_2c^2 \end{pmatrix}. \tag{41}$$

This result is almost the same as Korenman's [8], and differs only by the fact that the kinetic terms have different masses—a consequence of the fact that we have chosen to view the

shifted masses m_a as fundamental. As seen below, this only gives a higher order correction which can be neglected in the non-relativistic limit. The interaction Hamiltonian is:

$$H_{int} = \begin{pmatrix} 0 & \frac{\gamma \cdot F}{2c^2\sqrt{m_1 m_2}} \\ \frac{\gamma \cdot F}{2c^2\sqrt{m_1 m_2}} & 0 \end{pmatrix}, \tag{42}$$

After some algebra the Hamiltonian can be written in the following form:

$$\begin{aligned} H = & \left(-\frac{\hbar^2\nabla^2}{2\overline{m}} + \overline{m}c^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\hbar\omega_{12}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ & + \frac{\gamma \cdot F}{2c^2\sqrt{m_1 m_2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar\omega_{12}}{2}\begin{pmatrix} \frac{\hbar^2\nabla^2}{2\overline{m}m_1c^4} & 0 \\ 0 & \frac{-\hbar^2\nabla^2}{2\overline{m}m_2c^4} \end{pmatrix} \end{aligned} \tag{43}$$

where we have defined the energy difference $\omega_{12} = (m_1 - m_2)c^2/\hbar$.

The first three terms have a very natural physical interpretation: The first term is the free Hamiltonian for the "collective modes" of the atoms (the term proportional to the mass is just a shift in the energy and not relevant), while the next two describe the energy splitting and the corresponding dipole interaction with the electromagnetic field. Assuming that $\hbar\omega_{12} << \overline{m}c^2$, the last term is an order λ correction to the second term. It is therefore higher order in the non-relativistic expansion and consistency demands that we neglect it.

We drop the term in the Hamiltonian that corresponds to the collective modes and write the remaining piece as the sum of two terms:

$$H = \frac{1}{2}\hbar\omega_{12}\sigma_z + \frac{\gamma \cdot F}{2c^2\sqrt{m_1 m_2}}(\sigma_+ + \sigma_-) \tag{44}$$

where σ_z is the third Pauli matrix and σ_+ and σ_- are the usual raising and lowering operators for the atomic states:

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{45}$$

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{46}$$

We can now make contact with the Jaynes-Cummings model by assuming a single mode electric field, linearly polarized in the x-direction, as would be found in a high-Q cavity of volume V, for example. In terms of the standard harmonic oscillator creation and annihilation operators, the field can be written [13]:

$$\vec{E} = \hat{x}\mathcal{E}_\Omega(a + a^\dagger)\sin Kz \tag{47}$$

where $\mathcal{E}_\Omega = [\hbar\Omega/\epsilon_0 V]^{\frac{1}{2}}$ is the "electric field per photon" for an electric field of frequency Ω. In the above, $\hat{z}$ points along the longitudinal axis of the cavity and $K = \Omega/c$ is the magnitude of the corresponding wave number.

The interaction Hamiltonian now takes the form

$$H_{int} = -\frac{\gamma_{0i}E^i}{c^3\sqrt{m_1 m_2}}(\sigma_+ + \sigma_-)(a + a^\dagger) \tag{48}$$

$$= -\frac{\gamma_x\mathcal{E}_\Omega}{c^3\sqrt{m_1 m_2}}\sin Kz(\sigma_+ + \sigma_-)(a + a^\dagger) \tag{49}$$

where γ_x is the component of γ_{0i} in the direction of the electric field.

We can drop terms proportional to $\sigma_- a$ and $\sigma_+ a^\dagger$. These terms correspond to the simultaneous lowering of an atom and absorption of a photon, and the simultaneous raising of an atom and production of a photon, and we expect them to be suppressed. We can see that this is the case by looking at the evolution of the operators in the Heisenberg picture. Writing

$$\sigma_\pm(t) = \sigma_\pm(0)e^{\pm i\omega t}$$

$$a(t) = a(0)e^{-i\Omega t}$$

$$a^\dagger(t) = a^\dagger(0)e^{i\Omega t}$$

we find that $\sigma_- a$ and $\sigma_+ a^\dagger$ are proportional to $e^{\pm i(\omega+\Omega)t}$ and the other two products are proportional to $e^{\pm i(\omega-\Omega)t}$. We are interested in a system that is tuned close to resonance $\Omega \approx \omega$ and therefore, in the random phase approximation, terms proportional to $e^{\pm i(\omega+\Omega)t}$ will average to zero because of the rapid oscillation of the phase. The final result has precisely the form of the interaction term for the Jaynes-Cummings model:

$$H_{int} = \hbar g(a\sigma_+ + a^\dagger \sigma_-) \tag{50}$$

and we identify the Rabi frequency in our model as:

$$g \equiv -\frac{\gamma_x \mathcal{E}_\Omega \sin Kz}{\hbar c^3 \sqrt{m_1 m_2}} \tag{51}$$

This corresponds to the usual Rabi frequency [13]

$$g_R = -\frac{\langle e\vec{x}\rangle_x \mathcal{E}_\Omega}{\hbar} \sin Kz \tag{52}$$

on making the identification:

$$\gamma_x = \langle e\vec{x}\rangle_x \sqrt{m_1 m_2} c^3 \tag{53}$$

which is consistent with the identification for γ made in the previous section (eqn. (17)), apart from terms of order $(m_1 - m_2)/m$.

IV. THE EFFECTIVE ACTION AND ITS INTERPRETATION

Having established a connection to the Jaynes-Cummings model, we no longer need to refer to it and we can focus entirely on the relativistic case. Quantum corrections to the relativistic model may be computed using standard field theoretical prescriptions. The effective action is a particularly elegant way of generating such corrections. Although our theory is already an effective theory, this does not invalidate the procedure of looking for corrections due to correlations in our chosen field variables. If such corrections were already accounted for, they would simply renormalize away trivially in a renormalizable theory. In a non-renormalizable theory, this is not possible by definition and thus the corrections are always of interest. The calculation of loop corrections in non-renormalizable field theories has been shown to give accurate physics. See ref. [14] for a discussion of this.

The effective action is a generating functional for one-particle irreducible quantum corrections. It describes completely the behaviour of the averaged field (or background field) after all quantum fluctuations have been taken into account. If one defines the functional integral for a field Φ by

$$W[J] = -i \ln \int d\mu[\Phi] \exp\left\{ iS[\Phi] + \int dV \; J\Phi \right\} \tag{54}$$

where $\hbar = c = \mu_0 = \epsilon_0 = 1$, and the c-number average field by

$$\overline{\Phi} = \langle \Phi \rangle = \frac{\delta W}{\delta J} \tag{55}$$

then the effective action is given by the Legendre transform of $W[J]$ which displaces the explicit dependence on the source of fluctuations J, in favour of a dependence on the average field itself.

$$\Gamma[\overline{\Phi}] = W[J] - \int dV \; J\overline{\Phi} \tag{56}$$

The resulting object is in all senses an action for the average field. In the remainder of the paper we make use of the background field method to compute the effective action. We begin by dividing the field into an average part and a fluctuating part for convenience:

$$\phi_a \sim \overline{\phi}_a + \varphi_a, \tag{57}$$

where $\overline{\phi}_a$ is the average field and φ_a is the quantum field which replaces the total ϕ as the variable of integration in eqn (54). This division may now be used as a basis for generating a perturbation expansion for the effective action. Our model for the laser contains two fields ϕ_a and A_μ. We shall assume that the average external field $F_{\mu\nu} = 0$, so that $\overline{A}_\mu = \frac{\delta W}{\delta J_\mu}$ may always be gauged to zero in all physical results, provided only that the systems lives in a box with a simple topology. It is nevertheless useful to keep this quantity non-zero when using $\Gamma[\overline{\phi}, \overline{A}_\mu]$ as a generating functional, since derivatives with respect to the vector field allow us to easily calculate physical quantities of interest. We expand the action around these background fields

$$S\left[\overline{\phi}_a + \varphi_a, \overline{A}_\mu + A_\mu\right] = S_{\text{class}}[\overline{\phi}_a, \overline{A}_\mu] + S_2[\varphi_a, A_\mu] + S_{\text{int}}[\overline{\phi}_a, \varphi_a, A_\mu], \tag{58}$$

where S_{class} is the term composed purely of background fields, S_2 is quadratic in the quantum field variables and S_{int} is the remainder. The effective action is then given by the one-particle irreducible part of

$$\Gamma[\overline{\phi}_a, \overline{A}_\mu] = S[\overline{\phi}_a, \overline{A}_\mu] - i \ln \int d\mu[\varphi_a, A_\mu] e^{iS_{\rm int}} e^{iS_2}$$
$$= S[\overline{\phi}_a, \overline{A}_\mu] + \langle S_{\rm int} \rangle + i \langle (S_{\rm int})^2 \rangle + \ldots \tag{59}$$

We compute the effective action in two stages. First we consider fluctuatations in the photon field leading to an intermediary effective action Γ_A. These can be dealt with exactly and this leaves us with a result for the dynamics of the atomic system with all photon degrees of freedom eliminated. This is action could then be used to describe the situation in the micro-cavity maser where the measurable degree of freedom are the atomic states, and the effects of the photons are only felt indirectly. Secondly, we consider fluctuations in the atomic degrees of freedom, such as one would expect in a gaseous or solid state laser. This gives us the full effective action $\Gamma[\overline{\phi}_a]$.

We begin by considering the radiation field with a Lorentz gauge fixing term added and associated Lagrange multiplier $1/\alpha$,

$$S_M = \int dV_x \left\{ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + P^{\mu\nu} F_{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right\}. \tag{60}$$

Ghost terms may be absorbed into the functional measure in view of the trivial nature of the gauge field contribution.

The functional integral over A_μ may be performed immediately since it is Gaussian. Integrating by parts and shifting the quantum gauge field (the field of integration) $A_\mu \rightarrow A_\mu - 2\partial^\nu P_{\mu\nu}$, one obtains without modification to the functional measure,

$$S_M = \int dV_x \left\{ \frac{1}{2} A^\mu \left[-\Box \, \delta_\mu{}^\nu + \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial^\nu \right] A_\nu - 2 \int dV_{x'} (\partial_\mu P^{\mu\nu}) D_{\nu\sigma} (\partial_\rho P^{\rho\sigma}) \right\} \tag{61}$$

The integral over the gauge field A_μ is now a Gaussian and may be dealt with by standard results. This results only in a constant addition to the effective action which may be renormalized away by a shift of the arbitrary zero point for the energy scale. The result is the one-loop correction

$$\Gamma_A^{(1)}[P^{\mu\nu}] = \text{const} + 2 \int dV_x dV_{x'} \, (\partial_\mu P^{\mu\nu}) D_{\nu\sigma} (\partial_\rho P^{\rho\sigma}) \tag{62}$$

where the free photon Green function is defined by the relation

$$\left[-\Box\, g_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right] D^{\nu\rho} = \delta_\mu{}^\rho \delta(x, x'). \tag{63}$$

Using this general result we obtain the first stage effective action for atomic field ϕ_a:

$$\Gamma_A[\phi] = \int dV_x \left\{\frac{1}{2}\phi_a\left[-\Box + m_a^2\right]\phi_a + 2\int dV_{x'}\phi_a(x)\phi_b(x)\overline{V}^{abcd}(x, x')\phi_c(x')\phi_d(x')\right\} \tag{64}$$

and

$$\overline{V}^{abcd}(x, x') = \gamma_{ab}^{\mu\nu}\gamma_{cd}^{\rho\sigma}(\overset{x}{\partial}_\mu\overset{x'}{\partial}_\rho\, D_{\nu\sigma}(x, x')). \tag{65}$$

where we have introduced the short hand notation $\gamma_{ab}^{\mu\nu} = \gamma^{\mu\nu}\overline{\epsilon}_{ab}$.

To generate the second stage effective action, we expand the atomic variables about a background or external field. The physical significance of this step is the presence of measurable averages for the atomic variables in our system.

Expanding around free fields and dealing with the interaction term as an expansion of the exponentiated action, we get,

$$\Gamma[\overline{\phi}] = \Gamma_A[\overline{\phi}_a] - i\ln\int d\mu[\varphi_a]\left(e^{iS_{\rm int}}\right)\exp\left\{iS^{(2)}\right\} \tag{66}$$

$$= \langle S_{\rm int}\rangle + i\langle(S_{\rm int})^2\rangle + \ldots O(\overline{\phi}^3) \tag{67}$$

The Feynman (time-ordered) propagator is defined by

$$\langle\varphi_a(x)\varphi_b(x')\rangle = -iG_{ab}\delta(x, x'). \tag{68}$$

For the renormalizable $\overline{V}$ vertex we now obtain the part of the effective action which is quadratic in the background fields. The outstanding terms do not contribute to the self energy and therefore to the decay rates of the atomic levels.

$$\begin{aligned}\langle S_{\rm int}\rangle = -i\int dV_x dV_{x'}\overline{V}^{abcd}(x, x')\Big\langle&\overline{\phi}_a(x)\overline{\phi}_d(x')G_{bc}(x, x') + \overline{\phi}_a(x)\overline{\phi}_c(x')G_{bd}(x, x')\\ &+ \overline{\phi}_b(x)\overline{\phi}_c(x')G_{ad}(x, x') + \overline{\phi}_b(x)\overline{\phi}_d(x')G_{ac}(x, x')\Big\rangle + \text{disconnected}\end{aligned} \tag{69}$$

The matrix $\gamma_{ab}^{\mu\nu}$ is off-diagonal but symmetrical in a, b, so we may write

$$\overline{V}^{abcd} = \left\{\overline{V}^{1212} = \overline{V}^{2112} = \overline{V}^{2121} = \overline{V}^{1221} = \gamma^{\mu\nu}\gamma^{\rho\sigma}(\overset{x}{\partial}_\mu\overset{x'}{\partial}_\rho\, D_{\nu\sigma}(x, x'))\right\}. \tag{70}$$

This last result gives us

$$\langle S_{\rm int}\rangle = -4i\gamma^{\mu\nu}\gamma^{\rho\sigma}\int dV_x \int dV_{x'}(\overset{x}{\partial}_\mu \overset{x'}{\partial}_\rho\, D_{\nu\sigma}(x,x')) \\ \times\left\{\overline{\phi}_1(x)\overline{\phi}_1(x')G_{22}(x,x') + \overline{\phi}_2(x)\overline{\phi}_2(x')G_{11}(x,x')\right\}. \tag{71}$$

We are interested in the effective coupling constants of the quantized theory, which may be defined through derivatives of the effective action. These provide us with information about the decay rates or lifetimes of the atomic levels and corrections to the Rabi-flopping frequency. The momentum-space structure of these quantites also illustrate how photon energies are related to the interatomic spacings etc. Specifically, we wish to compute the diagonal scalar self-energy Σ_{aa}, whose imaginary part gives an indication of the decay rates of the levels,

$$\Sigma_{aa}(x,x') = \frac{\delta^2\Gamma[\overline{\phi}]}{\delta\overline{\phi}_a(x)\delta\overline{\phi}_a(x')}, \tag{72}$$

the interaction vertex (or generalized coupling constant)

$$\Gamma_\mu(x,x',x'') = \overline{\epsilon}_{ab}\frac{\delta^2\Gamma[\overline{\phi},\overline{A}_\lambda]}{\delta\overline{\phi}_a(x)\delta\overline{\phi}_b(x')\delta\overline{A}^\mu(x'')} \tag{73}$$

and the photon self-energy or polarization tensor

$$\Pi_{\mu\nu}(x,x') = \frac{\delta^2\Gamma[\overline{\phi},\overline{A}_\lambda]}{\delta\overline{A}^\mu(x)\delta\overline{A}^\nu(x')}. \tag{74}$$

V. ONE-LOOP RENORMALIZABILITY

We now verify the one-loop renormalizability of our theory. For the remainder of the paper, we choose natural units in which $\hbar = c = \epsilon_0 = \mu_0 = 1$. General arguments indicate that renormalizablity is connected with power-counting, or the dimension of the coupling constant. In this scheme there is only one scale of dimensions. Length and time are completely equivalent and mass is the inverse of length. A dimensional analysis of the action in these units leads to the conclusion that both the scalar and vector field ϕ_a and A_μ in $n+1$ space-time dimensions has engineering dimension

$$[\phi_a] = [A_\mu] = L^{\frac{1-n}{2}}. \tag{75}$$

A renormalizable quantum field theory is one in which all the infinities accrued by the calculational procedure can be defined away by reinterpreting the coupling constants appearing in the action. This is possible only if the infinite terms are of the same form as the original terms in the action which contain the coupling constants. In a non-renormalizable theory, it is not possible to absorb all infinities with a finite number of redefinitions.

For the moment we will consider both the non-renormalizable and renormalizable interactions, in order to contrast them:

$$\begin{aligned} \tilde{P}_{\mu\nu}: \quad & [\gamma^{\mu\nu}] = L^{\frac{n-1}{2}} \\ P_{\mu\nu}: \quad & [\gamma^{\mu\nu}] = L^{\frac{n-3}{2}} \end{aligned} \tag{76}$$

We may consider the case of both two and three spatial dimensions, since a laser often has an axial symmetry which reduces its effective dimensionality. For the first of the interactions in eqn. (76) $\gamma^{\mu\nu}$ has the dimensions of L in $3+1$ dimensions and $L^{\frac{1}{2}}$ in $2+1$ dimensions. In both cases the interaction is non-renormalizable. The second interaction is more successful. In $3+1$ dimensions, $\gamma^{\mu\nu}$ is dimensionless which implies that the theory is strictly (also called marginally) renormalizable. In $2+1$ dimensions, $\gamma^{\mu\nu}$ has the dimensions of $L^{-\frac{1}{2}}$, which implies that the theory is super-renormalizable.

Using the second interaction, the Jaynes-Cummings model can be represented as a renormalizable field theory given by the action in eqns. (2) and (3). We note that, although renormalizablity is often regarded as a critereon for choosing between field theories, it is not an infallible guide to their physicality. Quantum corrections to non-renormalizable theories are known to give accurate results in a number of cases [14]. Moreover, we have a natural energy cut-off for the kinetic motion of atoms, namely kT. Our primary reason for choosing the renormalizable interaction is that it is easier to calculate quantum corrections in this case; the lack of an explicit time-derivative preserves Lorentz covariance.

A. Scalar self-energy

We seek to calculate the one-loop self energy $\Sigma(p)$ and vertex function $\Gamma_\mu(p, p', q)$ and show that these terms have infinite pieces that have the same form as the original interaction, and thus can be reabsorbed into the coupling constants, the masses, and rescaling factors. We use cutoff regularization since we ultimately want to use our model to study laser physics, which will involve the imposition of boundary conditions. We define the subtraction scheme by expanding around the mass shell. We expand in $\frac{\delta}{M^2}$ where $\delta = m_1^2 - m_2^2$ and $M^2 = \frac{1}{2}(m_1^2 + m_2^2)$.

The bare theory gives a propagator of the form,

$$iG_{aa}(p) = \frac{i}{p^2 + m_{a0}^2 - i\epsilon}$$

which has a pole at $p^2 = -m_{a0}^2$. We calculate the polarization tensor and use the Dyson equation to obtain a propagator of the form,

$$iG_{aa}(p) = \frac{i}{p^2 + m_{a0}^2 + \Sigma_{aa}(p) - i\epsilon}$$

We define $\Delta m_a^2 = m_a^2 - m_{a0}^2$ and write,

$$iG_{aa}(p) = \frac{i}{p^2 + m_a^2 + [\Sigma_{aa}(p) - \Delta m_a^2] - i\epsilon}$$

We choose

$$\Delta m_a^2 = \Sigma_{aa}(-m_a^2) \tag{77}$$

so that the pole occurs at $p^2 = -m_a^2$ which we call the physical mass. Thus, the propagator can be written,

$$iG_{aa}(p) = \frac{i}{p^2 + m_a^2 + [\Sigma_{aa}(p^2) - \Sigma_{aa}(-m_a^2)] - i\epsilon}$$

$\Sigma_{aa}(p)$ is divergent and a requirement for renormalizabilty is that we can write (after regularization)

$$\Sigma_{aa}(p^2) - \Sigma_{aa}(-m_a^2) = (p^2 + m_a^2) f(\Lambda) \tag{78}$$

so that the propagator becomes

$$iG_{aa}(p) = \frac{iZ_{\phi_a}}{p^2 + m_a^2 - i\epsilon}; \quad Z_{\phi_a}^{-1} = 1 + f(\Lambda) \tag{79}$$

These redefinitions are equivalent to the statement that we can add counterterms to the Lagrangian of the form,

$$\mathcal{L}_{ct}^1 = \frac{1}{2}(Z_{\phi_a}^{-1} - 1)((\partial_\mu \phi_a)^2 + m_a^2 \phi_a^2) + \frac{1}{2}\Delta m_a^2 \phi_a^2$$

and absorb the infinities from the self energy $\Sigma_{aa}(p)$ in the mass shift Δm_a^2 and the wave-function scaling factor Z_{ϕ_a}.

In a similar way, the interaction part of the Lagrangian $\mathcal{L}_{int}$ gives rise to a bare vertex of the form,

$$\Gamma_\mu^{(0)} = 2\epsilon_{ab} q_{\mu'} \gamma_0^{\mu'\mu}$$

where q is the incoming photon momentum and $\gamma_0^{\mu\mu'}$ is the bare coupling constant. The one loop contribution to this vertex is divergent. We isolate the divergent part by performing a subtraction at the mass shell,

$$\Gamma_\mu^{(1)}(p, p', q) = \Gamma_\mu^{(1)}\Big|_{ms} + \tilde{\Gamma}_\mu^{(1)}$$

where the subscript ms means that the external momenta are evaluated on the mass shell, and $\tilde{\Gamma}$ is finite. A requirement of renormalizability is that we can write, after regularization,

$$\Gamma_\mu^{(1)}\Big|_{ms} = \Gamma_\mu^{(0)}(Z_1^{-1} - 1) \tag{80}$$

which means that we can absorb the infinite part of the one loop vertex graph into a redefinition of the coupling constant. This redefinition is equivalent to adding to the Lagrangian a counterterm of the form

$$\mathcal{L}_{ct}^2 = (Z_1^{-1} - 1)\mathcal{L}_{int}$$

In this section, we will calculate the one loop self energy and the one loop vertex function and use (77), (78), (79) and (80), to determine Δm_a^2, Z_{ϕ_a} and Z_1. We start from the following expression for the self-energy of the scalar field ϕ_1:

$$\Sigma_{11} = -4i\gamma^{\tau\mu}\gamma^{\lambda}_{\ \mu}\int \frac{d^4k}{(2\pi)^4}\frac{k_\tau k_\lambda}{(k^2-i\epsilon)((k+p)^2+m_2^2-i\epsilon)}$$

where we have used the Feynman gauge $\alpha = 1$ for the internal photon propagator. The self-energy for the field ϕ_2 will depend on m_1 in the same way. We rewrite the denominator using the usual Feynman parameter formula,

$$\frac{1}{k^2-i\epsilon}\frac{1}{(k+p)^2+m_2^2-i\epsilon} = \int_0^1 dx \frac{1}{(k^2(1-x)+[(k+p)^2+m_2^2]x-i\epsilon)^2}$$

We complete the square in the denominator and shift the integration variable $k = l - px$ to obtain,

$$\Sigma_{11} = -4i\gamma^{\tau\mu}\gamma^{\lambda}_{\ \mu}\int \frac{d^4l}{(2\pi)^4}\int_0^1 dx \frac{l_\tau l_\lambda + x^2 p_\tau p_\lambda}{(l^2+a^2-i\epsilon)^2}$$

where $a^2 = m_2^2 x + p^2 x(1-x)$ and we have dropped the terms linear in l which give zero by symmetric integration. We do a Wick rotation so that the integration contour lies along the imaginary axis and make the change of variable, $l_0 = il_4$ to obtain the Eucledian space integral,

$$\Sigma_{11} = 4\gamma^{\tau\mu}\gamma^{\lambda}\ _{\mu}\int \frac{d^4l}{(2\pi)^4}\int_0^1 dx \frac{l_\tau l_\lambda + x^2 p_\tau p_\lambda}{(l^2+a^2)^2} \tag{81}$$

The integral is infinite and we use cut-off regularization to render it finite. After regularization we can switch the order of integration and perform the l integration first. We consider the two pieces separately. First we evaluate the term proportional to $l_\tau l_\lambda$ and call it Σ_{11}^I. Under the integral sign we can replace $l_\tau l_\lambda$ by $\frac{1}{4}g_{\tau\lambda}l^2$ (by symmetric integration) which gives,

$$\Sigma_{11}^I = \gamma^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2+a^2)^2}$$

where $\gamma^2 = \gamma_{\mu\nu}\gamma^{\mu\nu}$. Doing the l integration gives,

$$\Sigma_{11}^I = \frac{\gamma^2}{(2\pi)^3}\int_0^1 dx(\Lambda^2 + a^2 - 2a^2 \ln\frac{\Lambda^2}{a^2})$$

We expand in $b = \delta/M^2$ and take only the leading order term. We calculate $\Sigma^I_{11}(-m_1^2)$ and $\Sigma^I_{11}(p^2) - \Sigma^I_{11}(-m_1^2)$. The result is,

$$\Sigma^I_{11}(-m_1^2) = \frac{\gamma^2}{(2\pi)^3}(\Lambda^2 - \frac{2}{3}M^2 \ln\frac{\Lambda^2}{M^2} - \frac{M^2}{9})$$
$$\Sigma^I_{11}(p^2) - \Sigma^I_{11}(-m_1^2) = -\frac{\gamma^2}{(2\pi)^3}[\frac{25}{18}(p^2 + m_1^2) + \frac{1}{3}(p^2 + m_1^2)\ln\frac{\Lambda^2}{M^2}] \tag{82}$$

Next we have to calculate the term proportional to $p_\alpha p_\beta$. From (81) we have,

$$\Sigma^{II}_{11} = 4\gamma^2_{\tau\lambda} q^\lambda q^\tau \int_0^1 dx\ x^2 \int \frac{d^4 l}{(2\pi)^4}\frac{1}{l^2 + a^2}$$

where $\gamma^2_{\tau\lambda} = \gamma^\mu_{\ \tau}\gamma_{\mu\lambda}$. Doing the l integration we obtain,

$$\Sigma^{II}_{11}(p) = \frac{4\gamma^2_{\lambda\tau}}{(2\pi)^3} p^\tau p^\lambda \int_0^1 dx\ x^2(\ln\frac{\Lambda^2}{a^2} - 1)$$

which gives,

$$\Sigma^{II}_{11}(-m_1^2) = \frac{4\gamma^2_{\tau\lambda}}{(2\pi)^3} p^\lambda p^\tau [\frac{1}{3}\ln\frac{\Lambda^2}{M^2} - \frac{1}{9}]$$
$$\Sigma^{II}_{11}(p^2) - \Sigma^{II}_{11}(-m_1^2) = -\frac{2\gamma^2_{\tau\lambda}}{3(2\pi)^3} p^\tau p^\lambda [\frac{p^2 + m_1^2}{M^2}] \tag{83}$$

Thus, from (77), (78), (79), (82) and (83) we obtain,

$$\Delta m_a^2 = \frac{\gamma^2}{(2\pi)^3} M^2 [\frac{\Lambda^2}{M^2} - \frac{2}{3}\ln\frac{\Lambda^2}{M^2} - \frac{1}{9} + 4\frac{\gamma^2_{\tau\lambda} p^\tau p^\lambda}{\gamma^2 M^2}[\frac{1}{3}\ln\frac{\Lambda^2}{M^2} - \frac{1}{9}]] \tag{84}$$
$$Z^{(-1)}_{\phi_1} = 1 - \frac{\gamma^2}{(2\pi)^3}[\frac{25}{18} + \frac{1}{3}\ln\frac{\Lambda^2}{M^2} + \frac{2}{3}\frac{\gamma^2_{\tau\lambda} p^\tau p^\lambda}{\gamma^2 M^2}] \tag{85}$$

and $Z_{\phi_2} = Z_{\phi_1}$. Keeping only the divergent terms, $Z^{(-1)}_{\phi_a}$ is a wavefunction renormalization factor of the usual form, and the first two terms in Δm_a^2 give an infinite shift in the mass term in the standard way. The fourth term in the expression for Δm_a^2 corresponds to a new interaction in the Lagrangian at the one loop level of the form,

$$\gamma^2_{\mu\nu}(\partial^\mu \phi)(\partial^\nu \phi)$$

B. Interaction vertex

Next we obtain the vertex renormalization constant from the one loop vertex correction shown in Fig XX. We obtain,

$$\Gamma^{(1)}_{\mu} = 8i\gamma^{\alpha}_{\lambda}\gamma^{b\lambda}\gamma^{\mu'\mu}q_{\mu'}\int\frac{d^4k}{(2\pi)^4}\frac{k_\alpha k_b}{(k^2-i\epsilon)((p'-k)^2+m_1^2-i\epsilon)((p-k)^2+m_2^2-i\epsilon)}$$

We rewrite the integral in terms of two Feynman parameters by using the expression,

$$\frac{1}{ABC} = 2\int_0^1 x\ dx\int_0^1 dy\frac{1}{(xyA+x(1-y)B+(1-x)C)^3}$$

with

$$A = (p-k)^2+m_2^2-i\epsilon$$

$$B = (p'-k)^2+m_1^2-i\epsilon$$

$$C = k^2-i\epsilon.$$

Putting the external scalars on the mass shell we obtain,

$$\Gamma^{(1)}_{\mu} = 16i\gamma^{\alpha}_{\lambda}\gamma^{\beta\lambda}\gamma^{\mu'\mu}q_{\mu'}\int\frac{d^4k}{(2\pi)^4}\int_0^1 x\ dx\int_0^1 dy\frac{k_\alpha k_\beta}{([k-(pxy+p'x(1-y)]^2+b^2-i\epsilon)^3}$$

where

$$b^2 = x^2m_2^2+\delta x^2(1-y)+x^2y(1-y)q^2$$

We shift the integration variable

$$l = k-(pxy+p'x(1-y))$$

and drop the terms linear in l which give zero by symmetric integration. We perform a Wick rotation so that the integration contour lies along the imaginary axis, and make the change of variable $k_0 = ik_4$. The result is,

$$\Gamma^{(1)}_{\mu} = -16\gamma^{\alpha}_{\lambda}\gamma^{\beta\lambda}\gamma^{\mu'\mu}q_{\mu'}\int\frac{d^4l}{(2\pi)^4}\int_0^1 x\ dx\int_0^1 dy\frac{l_\alpha l_\beta+M_{\alpha\beta}}{(l^2+b^2)^3}$$

where,

$$M_{\alpha\beta} = (p_\alpha xy + p'_\alpha x(1-y))(p_\beta xy + p'_\beta x(1-y))$$

We consider separately the terms proportional to $l_\alpha l_\beta$ and $M_{\alpha\beta}$. We will first do the integral for the term containing $l_\alpha l_\beta$ and call it $\Gamma^{(1)I}_\mu$. This term is divergent, and we use cutoff regularization. By symmetric integration we can write $l_\alpha l_\beta = \frac{1}{4} g_{\alpha\beta} l^2$ under the integral sign. Switching the order of integration and performing the l integration gives,

$$\Gamma^{(1)I}_\mu = -4\gamma^2 \gamma^{\mu'\mu} q_{\mu'} \int_0^1 x \; dx \int_0^1 dy (\ln\frac{\Lambda}{b} - \frac{3}{4}).$$

We set $q^2 = 0$ and do the integrals over x and y to obtain,

$$\Gamma^{(1)I}_\mu = -\frac{2}{(2\pi)^3}\gamma^2 \gamma^{\mu'\mu} q_{\mu'} [\ln\frac{\Lambda^2}{M^2} - \frac{1}{2}] \tag{86}$$

We calculate the term proportional to $M_{\alpha\beta}$ in the same way. Including this result, we obtain from (80) and (86),

$$Z_1^{-1} = 1 - \frac{\gamma^2}{(2\pi)^3}\left([\ln\frac{\Lambda^2}{M^2} - \frac{1}{2}] + \frac{2\gamma^2_{\alpha\beta}}{\gamma^2}[\frac{p'^\alpha p'^\beta}{M^2} - \frac{(p'^\alpha q^\beta + p'^\beta q^\alpha)}{2M^2} + \frac{q^\alpha q^\beta}{3M^2}]\right) \tag{87}$$

The terms in square brackets represent contributions from new interactions of the general form,

$$\gamma^{\alpha\lambda}\gamma^\beta_\lambda \gamma^{\mu\nu} \phi_a (\partial_\alpha \phi_b)(\partial_\beta \partial_\mu A_\nu).$$

This higher-derivative term could become important in the strong field limit and in non-perturbative regimes.

C. Photon polarization

Finally, we consider the photon polarization tensor. We have,

$$\Pi_{\mu\nu}(q) = -4iq_\alpha q_\beta \gamma^{\alpha\mu}\gamma^{\beta\nu} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m_1^2 - i\epsilon)((p+q)^2 + m_2^2 - i\epsilon)}$$

We separate the denominators using the Feynman parameter technique and perform a Wick rotation as before. The result is

$$\Pi_{\mu\nu}(q) = 4q_\alpha q_\beta \gamma^{\alpha\mu}\gamma^{\beta\nu}\int_0^1 dx \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+M^2)^2}$$

where

$$M^2 = m_1^2(1-x) + m_2^2 x + q^2 x(1-x)$$

Expanding around the mass shell, we isolate the divergent piece by setting $q^2 = 0$, which is equivalent to taking the first term in the expansion. The result is,

$$\Pi_{\mu\nu}(q) = 4q_\alpha q_\beta \gamma^{\alpha\mu}\gamma^{\beta\nu}\frac{1}{(2\pi)^3}[\ln\frac{\Lambda^2}{M^2} + 1]$$

which leads to an induced interaction of the form,

$$\frac{1}{4}\gamma^{\lambda\mu}\gamma^{\tau\nu}F_{\lambda\mu}F_{\tau\nu}$$

D. Renormalizability Revisited

In the introduction it was claimed that a dimensional coupling constant matrix $\gamma_{\mu\nu}$ ensured the renormalizability of the model. However, the above calculations show that the one loop divergences require counter terms of the form $\gamma^2_{\mu\nu}(\partial^\mu\phi)(\partial^\nu\phi)$ and $\frac{1}{4}\gamma^{\lambda\mu}\gamma^{\tau\nu}F_{\lambda\mu}F_{\tau\nu}$. These terms may be thought of as multiplicative modifications to the scalar field kinetic term in the Lagrangian and to the value of μ_0 and ϵ_0 in the Maxwell part. They arise because the orientation of the dipole $\gamma_{\mu\nu}$ breaks the rotational invariance of the theory, which is then reflected in the quantum corrections. We are not obliged to add counterterms of the form $\gamma^{\alpha\lambda}\gamma^\beta_\lambda\gamma^{\mu\nu}\phi_a(\partial_\alpha\phi_b)(\partial_\beta\partial_\mu A_\nu)$ since these new interactions yield finite results (at least to one-loop), but the appearence of such terms nevertheless indicates that they are an integral part of the structure of the relativistic theory and should therefore be considered too.

The question then arises: is the theory, as given, renormalizable or not? We point out that, in a renormalized field theory, it is the renormalized values of the parameters which

are to be identified with the physical constants in an experiment. In our case $\gamma_{\mu\nu}$ is to be indentified with the dipole moment of an atomic system. In fact, the one loop divergences simply tell us that there are additional, relativistically covariant terms that are second order in derivatives of the fields that we could have added to the classical action. These terms correspond to relative permittivities and permeabilities. These are the only such terms which need to be added, and with the addition of these terms, the model would indeed be fully renormalizable. However, note that these two terms involve γ^2, which we assume is small, so according to the assumptions on which we base our perturbative expansion they are probably negligable. Their physical significance is not redundant however: in the limit of large electromagnetic fields, very high kinetic energies and strong dipole couplings, these extra terms become significant and predict new physics to be identified with experiments. For the present paper, we take the pragmatic approach however and assume that such terms will not contribute signicantly. In effect we are renormalizing the new γ^2 couplings to zero. This is a significant improvement over the non-renormalizable choice of coupling, in which new, higher derivative interactions would appear at all orders in perturbation theory. In renormalization group philosophy, one would say that we are expanding our theory in a region of Lagrangian-space which is closer to a renormalization group fixed point.

VI. CONCLUSIONS

We have presented a relativistic model for the interaction of a two state atom with an electromagnetic field and verified that it reduces to the Jaynes-Cummings model in the appropriate limits. We have also shown how to compute higher order, quantum corrections and verified that the model is one-loop renormalizable. By identifying the renormalized value of $\gamma_{\mu\nu}$ with observed dipole moments, or the Rabi flopping frequency of known systems, we have a prescription for gauging the magnitude of corrections which lead to the onset of new physics. The decay rates of the atomic levels may be identified with imaginary contributions to the self-energy Σ_{aa}, for which we are able to calculate an explicit expression, rather than

merely a formal expression as in Korenman's work. In the present paper, we have been mostly concerned with the self-consistency of our proposed model and have presented only a zero-temperature expression for the self-energy. In future work we shall compute the finite temperature self-energy, where the natural cut off Λc^2 is of the order of kT and obtain a more accurate gauge of the decay rate by looking at retarded (causal) boundary conditions, rather than the Feynman boundary conditions used here. It will also be natural to look at non-equilibrium systems, and extend our ananlysis to non-linear phenomena where some of the assumptions made in this paper begin to falter.

Most laser systems are well described by non-relativistic physics. We consider the most important result of our paper to be the identification of a model which can be straightforwardly solved in real-space, with arbitrary boundary conditions, as well as in many-particle theories at finite temperature and non-equilibrium. The use of relativistic field theory simplifies calculations greatly compared to direct non-relativistic formulations. It also addresses quantum corrections at the level of the Lamb shift, where corrections are measurably significant in atomic systems [15,4], and removes some of the arbitrariness of previous work on lasers by tying laser physics to a model which can easily be be analysed within the framework of a renormalization group philosophy. This is significant because it indicates which results are independent of the specific details of microscopic theory one chooses to work with.

Our paper opens a doorway to the study of the statistical mechanics of photons and atoms in cavities and free space, a topic which we intend to pursue in later work. Interesting studies include the use of our model to study the micromaser with proper finite boundary conditions and partially reflecting surfaces, and in an expanding or contracting spherical cavity, as a toy model for light generation by bubbles in sonoluminescence, and porous silcon.

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